UNIWERSYTET WARSZAWSKI
Wydział Matematyki, Informatyki i Mechaniki
Instytut Matematyki Stosowanej i Mechaniki

Prof. Dr. hab. Urszula Foryś
chair of the Section of Biomathematics and Game Theory
Institute of Applied Mathematics and Mechanics
Faculty of Mathematics, Informatics and Mechanics
University of Warsaw
Banacha 2, 02-097 Warszawa

Święcice, 20.10.2025

REVIEW OF THE DOCTORAL DISSERTATION

Title of the doctoral dissertation: Discrete neuron models from the point of view of dynamical system theory

Author of the doctoral dissertation: mgr Frank Fernando Llovera Trujillo

Supervisor of the doctoral dissertation: dr hab. Piotr Bartłomiejczyk
Auxiliary supervisor: dr inż. Justyna Signerska-Rynkowska

The review was prepared in response to my appointment by the Council of the Scientific Field of Exact Sciences of the Gdańsk University of Technology on July 11, 2025, to act as a reviewer in the procedure for awarding the academic degree of doctor of exact and natural sciences in the discipline *Mathematics* to Mr. Frank Llovera Trujillo, M.A. The aim of the review is to assess whether the doctoral dissertation by Mr. Llovera Trujillo meets the conditions specified in Article 187 of the Act of 20 July 2018 – *Law on Higher Education and Science* (Dz. U. z 2020 r. poz. 85 z późn. zm.).

The doctoral dissertation "Discrete neuron models from the point of view of dynamical system theory" concerns the mathematical description of the action of single neurons using one-dimensional discrete dynamical systems, in particular the study of the properties of these systems. The dissertation consists of the following scientific publications:

- F. Llovera Trujillo, J. Signerska-Rynkowska, P. Bartłomiejczyk, "Periodic and chaotic dynamics in a map-based neuron model", *Math. Met. Appl. Sci.* 2023, **46**, 11906–11931;
- P. Bartłomiejczyk, F. Llovera Trujillo, J. Signerska-Rynkowska, "SPIKE PATTERNS AND CHAOS IN A MAP–BASED NEURON MODEL", *Int. J. Appl. Math. Comput. Sci.*, 2023, **33**(3), 395–408;
- P. Bartłomiejczyk, F. Llovera Trujillo, J. Signerska-Rynkowska, "Analysis of dynamics of a map-based neuron model via Lorenz maps", *Chaos,* 2024, **34**, 043110;
- F. Llovera Trujillo, "On computing periodic orbits itineraries for Lorenz-like maps", *Mathematica Applicanda*, 2024, **52**(2), 245–269.

1. Topics of the doctoral dissertation

The subject of the doctoral dissertation is reflected in the title of this dissertation. Each of the scientific publications included in the thesis refers to a certain one-dimensional discrete dynamical system, which is constructed as an approximation of a two-dimensional discrete model of the neuron action potential known from the literature. Such models are extremely interesting from the point of view of neuroscience, both as a description of a single neuron and as a basis for the description of more complex neural networks. The dissertation attempts to systematize the previously known properties of this type of models, as well as to prove additional theorems concerning general properties of Lorenz maps, β-transformations or unimodal maps with negative Schwarzian derivative, as well as specific theorems for the models discussed in the articles. Theoretical investigation is accomplished with numerical results illustrating various patterns that can be obtained for one-dimensional maps, also in comparison with two-dimensional original models. The series of articles is thematically related, presents new results on modeling neuronal dynamics and may constitute the basis for applying for a PhD degree.

2. Research objectives

Because a doctoral dissertation consists of a series of articles, I would expect coherent research objectives to be presented in the introduction. Unfortunately, the dissertation is provided with only a brief summary, so only the objectives of the individual articles within the dissertation can be discussed.

The first article "Periodic and chaotic dynamics in a map-based neuron model" is devoted to the Chialvo model. On the basis of this model its one-dimensional approximation is considered. This approximation forms a unimodal map with negative Schwarzian derivative. The aim was to study the model properties, including existence and stability of fixed points, bifurcation analysis and chaos recognition.

The next two articles focus on the Courbage *et al.* model, which is a bit different than the Chialvo model. The paper "Spike patterns and chaos in a map-based neuron model" was mostly related to the piecewise linear model (called 1D plCNV model), while in "Analysis of dynamics of a map-based neuron model via Lorenz maps", non-linear cubic map with additional parameter was considered. Here the theory of Lorenz-like maps and β-transformations was systematically used and suitably expanded to explain complex dynamics of these models. The main aim was to study periodic and chaotic dynamics, which was present in vast majority of the models parameters.

The fourth article presents a procedure that allows for numerical determining the itineraries of periodic orbits of Lorenz-like maps. Analytical investigation of such orbit can be extremely difficult or even impossible, so numerical procedures can be of great importance.

In my opinion, the research goals set are important, non-trivial, the methodology used is appropriate and up-to-date, thus meeting the requirements for doctoral theses.

3. Organization of the dissertation

The doctoral dissertation under consideration begins with a short summary, both in English and Polish.

The first article "Periodic and chaotic dynamics in a map-based neuron model" devoted to the Chialvo model was published in a recognizable international journal *Mathematical Methods in the Applied Sciences* (100 points), and is cited 14 times according to the Scholar Google. It is important to mark that the PhD candidate F. Llovera Trujillo is a first author of this paper. There are 6 sections, 2 appendices, 16 figures, 1 table and 46 bibliography positions. In INTRODUCTION, the original Chialvo model and its quasi-steady approximation called 1D Chialvo model were presented with the review on the literature related to the topic. Next section BASIC DEFINITIONS presented definition of S-unimodal map (that is a unimodal map with negative Schwarzian derivative), and short review on bifurcations in the context of such maps. Section DYNAMICAL CORE OF THE CHIALVO MAP was devoted to auxiliary analysis of 1D Chialvo model restricted to its dynamical core, that is its invariant domain. Next, in section PERIODIC VERSUS CHAOTIC BEHAVIOUR, these two types of oscillatory dynamics were studied. Analytical results were illustrated by numerical examples, first for 1D and also for 2D model. The main text ended with DISCUSSION. APPENDIX A consisted of the proofs related to the dynamical core of 1D map, while APPENDIX B was devoted to the proof of conditions for topological chaos.

The second article "Spike patterns and chaos in a map-based neuron model" devoted to the Courbage *et al.* model (called CNV model in the paper) was published in the *International Journal of Applied Mathematics and Computer Science* (140 points in 2023) and cited 7 times according to the Scholar Google. Unfortunately, there is no information about the contribution of F. Llovera Trujillo in preparation of this article. However, I can expect that this is similar as in the case of the next article, where this is clearly stated. There are 10 sections, 5 figures and 29 bibliography positions (I'd like to highlight that in the case of this issue there was a limitation to 15 pages per article). The first section INTRODUCTION was devoted to introducing the considered model and discussing its applications as well as some bibliography related to the model analysis. Next, in LORENZ-LIKE AND EXPANDING LORENZ MAPS, the definitions necessary for describing chaos (in the sens of Devaney) and its components were introduced, together with the concept of topological entropy, Lorenz-like maps and expanding Lorenz maps. Then, in ROTATION NUMBER AND INTERVAL, the concept of rotation numbers (which always exists for a periodic point) and rotation intervals was presented. In the next section FINITE UNIONS OF PERIODIC ORBITS AND FAREY-LORENZ PERMUTATIONS, for Lorenz-like maps, an idea of itineraries of periodic orbits together with know results on finite unions of periodic orbits and their permutations was described. L-permutations and Farey-Lorenz permutations were introduced. Section 5 LORENZ MAPS AND CHAOS presented results related to the elements of chaos for expanding Lorenz maps. Next Section 6 β-TRANSFORMATIONS, was devoted to piecewise linear maps called β-transformations. The definition, results related to the fixed and periodic orbits, invariant probability measures and chaos were presented. Section 7 ONE-DIMENSIONAL COURBAGE–NEKORKIN–VDOVIN MODEL presented the map g describing 1D CNV model and the basic assumptions posed for this map. In PERIODICITY AND CHAOS IN THE 1D plCNV MODEL, the theoretical results presented in the previous sections were applied to 1D CNV model, and then in INTERPRETATION OF ORBIT ITINERARIES IN TERMS OF SPIKING PATTERNS, these theoretical results served as an explanation for patterns observed in neuronal dynamics. The article ended with CONCLUSIONS section.

The third article "Analysis of dynamics of a map-based neuron model via Lorenz maps" devoted to the modification of the non-linear Courbage *et al.* model was published in *Chaos* (140 points) and cited 3 times according to the Scholar Google. It consisted of 9 sections and 6 appendices; there are 24 figures, 5 tables and 64 bibliography positions. INTRODUCTION section presented a more detailed review of discrete 2D models of neural activity, including the original CNV model, explaining the necessity of modifying the nonlinear model (adding additional parameter) and the rationale behind the 1D model analysis. Section ROTATION THEORY FOR LORENZ MAPS summarizeed necessary definitions and results related to the topic. Next, in TWO VERSIONS OF ONE DIMENSIONAL COURBAGE-NEKORKIN-VDOVIN MODEL, the models in consideration (piecewise linear and nonlinear) were presented, where 1D nlCNV model was of the main interest. Section ANALYSIS OF INVARIANT INTERVALS FOR 1D nlCNV MODEL described invariant intervals for the considered model depending on the parameters. Next, in CHAOTIC BEHAVIOUR IN THE 1D nlCNV MODEL section, first general conditions of chaotic behavior were presented, then conditions in α-β parameter plane were stated. Subsequent section PERIODICITY AND ROTATION INTERVAL FOR THE 1D nlCNV MODEL discussed fixed points, 2-periodic orbits and rotation intervals. ORBIT ITINERARIES AND COMPLEXITY OF SPIKE-TRAINS was the last section devoted to the model introduced at the beginning of this article. It presented theoretical and numerical results related to various types of patterns in 1D nlCNV model, while the next section ACTIVITY PATTERNS IN THE MODEL compared the dynamics of the model with constant and time-dependent input current. Obtained results were summarized and discussed in DISCUSSION

AND CONCLUSIONS section. APPENDIX A presented the definition and basic properties of Lorenz-like maps. APPENDICES B, C, D and E were devoted to the proofs of Theorems 1.5, 3.2, 4.3 and 5.2 respectively. APPENDIX F presented the theory of periodic orbits and their finite concatenations. It should be marked that numerical illustrations presented in this paper are very interesting and make this article more interesting from the neuroscience point of view.

The third paper differs from the previous ones in clear description of the authors contribution from which I can expect that the previous theoretical results were also obtained with equal contributions from all authors, while the numerical implementation was mainly handled by the PhD candidate Frank Llovera Trujillo. Moreover, this is consistent with the content of the last article, which was single-authored and was devoted to numerical methods.

The fourth article "On computing periodic orbits itineraries for Lorenz-like maps" was single-authored, as mentioned above, and was published in the journal of Polish Mathematical Society *Mathematica Applicanda* (20 points). It consisted of 7 sections, contains 8 figures, 4 tables and 14 bibliography positions. Introduction gave the motivation for undertaking this work. PRELIMINARIES presented basic definitions and tools necessary in the paper (Lorenz-like maps, rotation numbers and intervals, Farey sequences, binary sequences, essential classes). Next, in the section FUPO THEORY, another necessary notion was recalled and illustrated. Then, in ALGORITHMS, the core of the article was presented with subsequent algorithms proposed, leading to the main algorithm that allowed for finding rotation interval with optimal Farey neighbours inside and finite itineraries for them, which in turn served as a basis for the set of essential patterns. In the section EXAMPLES, the algorithms were implemented in MATLAB to illustrate their work for specific maps. The next section ANALYSIS OF THE SEQUENCES presented analysis of the complexity of patterns that can be obtained using proposed algorithms. The last section DISCUSSION briefly summarized the presented results.

4. Scientific achievements

The main scientific achievements on the basis of which I am requesting admission of Mr. Frank Llovera Trujillo to the PhD defense are:

1. <u>In the first article</u>

- Theorems 4, 5 (with respect to parameter r for fixed k) and 6 (with respect to parameter k for fixed r) describing bifurcations in 1D Chialvo model.
- Comparison between 1D and 2D model showing that the dynamics of 1D model can be even more complex than 2D.
- Theorems 7, 8 and 10 related to periodic attractors in 1D Chialvo model.
- Theorems 11 and 13 related to metric and topological chaos, respectively.

2. <u>In the second article</u>

- Proposition 1 showing that an expanding Lorenz map is expansive.
- Proposition 4 related to the number of periodic orbits of expanding Lorenz maps with periodic itinerary sequence.
- Proposition 5 showing that an expanding Lorenz map is transitive if and only if it is chaotic in the sense of Devaney.
- Theorem 7 describing conditions of chaos in the sense of Devaney for expanding Lorenz maps.
- Theorems 10 and 11 related to chaos in the sense of Devaney for β-transformations.
- Theorems 12, 13 and 14 describing the dynamics (fixed points, chaos, absolutely continuous invariant probability measure) of the map g reflecting 1D CNV model.
- Corollaries 3, 4 and 5 related to periodic patterns in 1D CNV model, and Examples 1 and 2 presenting specific patterns.

3. In the third article

- Theorem 1.5 presenting the condition for 2-periodic orbit existence for Lorenz-like maps.
- Theorem 3.2 related to the invariant intervals for 1D nlCNV model.
- Theorems 4.1, 4.3 describing conditions for chaos in 1D nlCNV model.
- Theorem 4.5 related to the unique ergodic invariant probability measure for 1D nlCNV model.
- Theorem 5.2 and numerical analysis devoted to 2-periodic orbits for 1D nlCNV model.
- Numerical analysis of rotation intervals for 1D nlCNV model.
- Numerical analysis of itineraries and finite unions of periodic orbits for 1D nlCNV model.
- Numerical results related to various types of patterns in 1D nlCNV model.
- Numerical results related to varying input current.

4. In the fourth article

- Algorithms 1, 2, 3 allowing for computation of rotation numbers and intervals.
- Algorithms 4, 5, 6 being auxiliary for finite sequences.
- Algorithms 7, 8, 9 leading to finding itineraries.
- Algorithm 10 generating the sequence of essential patterns of a given length.
- Algorithm 11 generating concatenations of two finite binary sequences.
- Algorithm 12 generating concatenation of itineraries of fupo.
- Analysis of the complexity of patterns that can be generated using proposed algorithms.

5. Critical remarks

Above I have already presented critical remarks regarding the lack of a clearly formulated hypothesis, as well as the editorial side of the dissertation. Below I present some more detailed remarks.

Detailed remarks:

- In the first article, Subsection 2.1, the authors defined *critical points* of a map f as such points at which its derivative equals 0. This can be misleading because, in general, in the theory of dynamical systems, critical points are also fixed points.
- In the first article, in Subsection 5.1, Theorem 9, the notion of *nondegenerate fixed point* is used without defining it. What can happen if such point is degenerate? Moreover, in Subsection 3.4, the authors used the notion of *superattracting fixed point*, again without definition.
- In the first article, at the beginning of Section 4, it is not clear why to restrict the analysis to the dynamics core. In my opinion, it would be better to have it explained in much more details, also with additional explanation what could be this core in the considered model.
- In the first article, Subsection 4.1, the map dynamics is not clearly described; especially the dependence of the dynamics on the model parameter should be highlighted and the interval on which the dynamics is studied should be pointed out.
- In the second article, in Introduction, it would be good to better explain why the posed assumption $m_0 < 1$ guarantees that the map is 1 to 1.
- In the second article, in the proof of Proposition 1, I found a typo: here is "and $f^n(x) \in [c, c + \delta_1]$" instead of $f^n(y)$.
- In the second article, in Subsection 4.3, the sentence "We will refer to P as the slow orbit and Q as the fast orbit" is not completely clear to me; namely, it is not clear how P and Q are defined.
- In the third article, it would be better to explain in more details the reason for exchanging the piecewise linear function F with a cubic polynomial. Is it important that it is cubic?
- In the fourth article, in Introduction, it would be good to have more detailed explanation why calculating rotation numbers/intervals and itineraries of Lorenz-like maps is important, especially in the context of the whole dissertation.

- In the fourth article, in Subsection 2.1, there appears a notion of *critical point* which is completely different from those in the first article, and again can be misleading in comparison to the common notion of critical points in the dynamical systems theory.
- In the fourth article, in Section 3, there is again the same problem with P and Q as in the third article.

6. Conclusion

In my opinion, the doctoral dissertation by M.A. Frank Llovera Trujillo contains valuable results that complement and organize the findings on the dynamics of Lorenz-like maps and β-transformations, and also demonstrates their application to models describing the functioning of neurons. This dissertation is a very good example of a doctoral thesis in applied mathematics, in which theoretical results are applied to specific models and illustrated with appropriate numerical simulations. The author's publication record is, of course, sufficient in light of the regulations.

Therefore, this dissertation meets the conditions specified in Article 187 of the Act of 20 July 2018 and may be the subject of a public defense in the field of exact and natural sciences in the discipline *Mathematics.*

I request that the doctoral dissertation of Mr. Frank Llovera Trujillo be admitted to public defense.

Urszula Foryś